

Exact Dragging of Inertial Axes by Cosmic Energy-Currents on the Past Light-Cone: Mach's Principle

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(Dated: June 19, 2014)

We prove exact rotational dragging of local inertial axes (\equiv spin axes of gyroscopes) by arbitrary cosmic energy-currents on the past light-cone of the gyroscope for linear perturbations of Friedmann-Robertson-Walker cosmologies. Hence, the principle formulated by Mach holds for arbitrary linear cosmological perturbations.

PACS numbers: 04.20.-q, 04.25.-g, 04.20.Cv, 98.80.Jk

I. THE HYPOTHESIS FORMULATED BY ERNST MACH

Experimentally spin axes of gyroscopes directly give the time-evolution of local inertial axes (as in inertial guidance systems). Conversely, relative to local inertial axes there is no gyroscope-precession. This is a *local* fact.

In a *cosmological* context, we have a super-precise *observational fact*: Spin-axes of gyroscopes do not precess relative to quasars, except for an extremely small dragging effect by Earth-rotation, the Lense-Thirring effect, which is negligible for gyroscopes away at a few Earth radii.

The question addressed in Mach's principle: *What physical cause* determines the time-evolution for spin-axes of gyroscopes, i.e. the time-evolution of inertial axes? In the words of John A. Wheeler: "Who gives the marching orders" to gyroscope axes (\equiv inertial axes)?

The *postulate formulated by Mach* [1]: Inertial axes exactly follow an average of the motion of cosmological masses: *exact frame-dragging*.

Since Newton's gravitational force cannot exert a torque on a gyroscope, Mach wrote: it is unknown, what new force could do the job.

In General Relativity, gravito-magnetism causes the Lense-Thirring effect, *extremely small torques* on gyroscopes in orbit around the Earth (caused by the Earth's rotation) detected by Gravity Probe B. In striking contrast, *Mach postulated exact dragging*, not a little bit of dragging.

Mach wrote that he did not know, *what average* of cosmological masses and their motions should be taken. I. Ciufolini and J.A. Wheeler wrote in 1995 [2] that it is *still unknown* what average should be taken.

II. RESULTS

In Refs. [3–5] we have proved exact dragging of inertial axes by cosmic energy-currents (Mach's principle)

on *space-like slices* (slices connecting points of equal local Hubble expansion-rate) for all possible linear perturbations of all Friedmann-Robertson-Walker (FRW) backgrounds.

Our new results: We prove exact dragging of local inertial axes at any space-time point P_0 by cosmic energy-currents on the *past light-cone* of the gyroscope-observation at P_0 for all possible linear perturbations of spatially flat FRW backgrounds.

The angular momentum constraint at P_0 from the *past light-cone* of P_0 for linear perturbations gives a linear *ordinary differential equation* in the radial variable.

The solution of the *angular momentum constraint* from the past light-cone gives *nothing more* and *nothing less* than (1) the proof of *exact dragging* of inertial axes by cosmic energy currents, (2) the form of the *dragging weight-functions* for various Hubble-rate histories.

III. PAST LIGHT-CONE COORDINATES FOR AN UNPERTURBED FRW UNIVERSE

In an unperturbed and spatially flat Friedmann-Robertson-Walker (FRW) universe, we single out one comoving observer, and we choose the spatial origin at his position. The comoving distance χ and the conformal time η are defined (with $c \equiv 1$),

$$\begin{aligned}\chi &\equiv \text{comov. distance from observ.} & \chi &\equiv r/a(t), \\ \eta &\equiv \text{conformal time,} & d\eta &\equiv dt/a(t),\end{aligned}\quad (1)$$

where r is the measured radial distance from the origin at fixed Hubble-time, dt is the measured time interval in a comoving frame, and $a(t)$ is the scale factor. The scale factor is set to one at $t_0 \equiv$ observation-time, $a(t_0) \equiv 1$. The light cones are at 45 degrees in the (η, χ) -plane, therefore (η, χ) is a conformal pair.

In the retarded Green function for a given conformal observation-time η at the spatial origin, the earlier conformal source-emission-time η' at comoving source-emission-distance χ' on the past light-cone of the observation is

$$\eta' = (\eta - \chi') \quad \text{with} \quad c \equiv 1.$$

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The conformal *past-light-cone coordinate* v is defined,

$$v \equiv \eta + \chi. \quad (2)$$

The coordinate v is constant on each of the past-light-cones of the chosen observer, i.e. v labels the past light-cones for our chosen observer. At the position of the observer, v is equal to the conformal time η . In the integration over sources for *retarded potentials*, v is fixed, and (χ, θ, ϕ) are the integration variables.

For an unperturbed and spatially flat FRW universe, the metric in past-light-cone coordinates (v, χ, θ, ϕ) is

$$ds^2 = a^2(\eta)_{\eta=v-\chi} [-dv^2 + 2 dv d\chi + \chi^2 d\omega^2],$$

where $d\omega^2$ is the line element on the unit 2-sphere,

$$d\omega^2 \equiv d\theta^2 + \sin^2 \theta d\phi^2.$$

The non-zero components of the unperturbed metric are

$$\begin{aligned} g_{vv}^{(0)} &= -a^2, & g_{v\chi}^{(0)} &= a^2, \\ g_{\theta\theta}^{(0)} &= (a\chi)^2, & g_{\phi\phi}^{(0)} &= (a\chi \sin \theta)^2, \\ (-\det g_{(0)})^{1/2} &\equiv (-g_{(0)})^{1/2} = a^4 \chi^2 \sin \theta. \end{aligned} \quad (3)$$

Note that $g_{\chi\chi}^{(0)} = 0$, because along a world line of a photon $ds^2 = 0$, and for a photon observed at $\chi = 0$, (v, θ, ϕ) is fixed, while $d\chi \neq 0$.

The inverse of the unperturbed metric is non-trivial only for the inverse of the (2x2)-matrix in the (v, χ) -sector. The nonzero elements of the unperturbed inverse metric are

$$\begin{aligned} g_{(0)}^{\chi\chi} &= \frac{1}{a^2}, & g_{(0)}^{v\chi} &= \frac{1}{a^2}, \\ g_{(0)}^{\theta\theta} &= \frac{1}{(a\chi)^2}, & g_{(0)}^{\phi\phi} &= \frac{1}{(a\chi \sin \theta)^2}. \end{aligned} \quad (4)$$

Note that $g_{(0)}^{vv} = 0$.

IV. VECTOR SPHERICAL HARMONICS $\vec{X}_{\ell m}^\pm$

Vector spherical harmonics form a basis for vector fields tangent to 2-spheres. They have been discussed in detail in Section IV of [5].

The vector spherical harmonics of *Regge and Wheeler* $\tilde{x}_{\ell m}^\pm$ [6] are defined by

$$\begin{aligned} \tilde{x}_{\ell m}^+ &\equiv dY_{\ell m} & \Leftrightarrow & (x_{\ell m}^+)_{\alpha} \equiv \partial_{\alpha} Y_{\ell m}, \\ \tilde{x}_{\ell m}^- &\equiv -^{(2)} * dY_{\ell m} & \Leftrightarrow & (x_{\ell m}^-)_{\alpha} \equiv -\varepsilon_{\alpha\beta} g^{\beta\gamma} \partial_{\gamma} Y_{\ell m}. \end{aligned} \quad (5)$$

On the left is the abstract notation of differential forms, on the right is the explicit component notation: $\tilde{x}_{\ell m}^+$ is the gradient of $Y_{\ell m}$, while $\tilde{x}_{\ell m}^-$ is its Hodge dual on the 2-spheres, denoted by $^{(2)}*$. On the 2-sphere of any radius, the Levi-Civita tensor is $\varepsilon_{\alpha\beta}$. The vector spherical

harmonics of Regge and Wheeler have covariant components (\equiv 1-form components) *independent of the radial coordinate* χ ,

$$\partial_{\chi}(x_{\ell m}^{\pm})_{\alpha} = 0, \quad \alpha = (\theta, \phi). \quad (6)$$

In contrast, the *physical vector spherical harmonics* $\vec{X}_{\ell m}^{\pm}$ used in classical electrodynamics [7] have a *point-wise norm* $g(\vec{X}_{\ell m}^{\pm*}, \vec{X}_{\ell m}^{\pm})$ *independent of the radial coordinate* χ , and therefore they have *LONB-components* (denoted by hats) *independent of the radial coordinate*,

$$\nabla_{\chi} \vec{X}_{\ell m}^{\pm} = 0, \quad \partial_{\chi} (X_{\ell m}^{\pm})_{\hat{k}} = 0. \quad (7)$$

The parity is $P = (-1)^{\ell}$ for $\vec{X}_{\ell m}^+$ (“natural parity”), while $P = (-1)^{\ell+1}$ for $\vec{X}_{\ell m}^-$ (“unnatural parity”).

As shown in the next section, the precession of a gyroscope at the origin can be caused only by cosmological energy currents with $\ell = 1$ and parity $P = +1$, i.e. with a superscript minus for the unnatural parity sequence. For a given source-radius r_s , in the Green function, we can specialize to $m = 0$ without loss of generality.

For $m = 0$, one has rotational symmetry around the z -axis, and the vector field $\vec{X}_{\ell, m=0}^-$ points in the ϕ -direction. If the vector field is a 3-velocity field, the LONB component $V_{\hat{\phi}}$ is the measured 3-velocity in the ϕ -direction, and the contravariant component V^{ϕ} is the measured angular velocity around the z -axis, $d\phi/dt$.

For $\ell = 1$ (with $m = 0, P = +1$), the angular velocity around the z -axis is independent of θ . This is a *rigid rotation* around the z -axis with angular velocity $\Omega = (d\phi/dt) = v^{\phi}$. Using $Y_{\ell=1, m=0} = \sqrt{3/(4\pi)} \cos \theta$,

$$\begin{aligned} \vec{X}_{\ell=1, m=0}^- &\equiv \sqrt{3/(4\pi)} \vec{V}, \\ V^{\phi} &= 1, & \text{angular velocity,} \\ V_{\hat{\phi}} &= \sin \theta, & \text{velocity.} \end{aligned} \quad (8)$$

For any (ℓ, m) , the unnatural-parity vector spherical harmonics $X_{\ell, m}^-$ are called *toroidal*.

V. PRECESSION OF GYROSCOPE-SPIN CAUSED BY TOROIDAL VORTICITY PERTURBATIONS WITH $\ell = 1$

We treat all linear perturbation fields on a spatially flat FRW background ($K = 0$), and *all* energy-momentum-stress tensors, i.e. all types of matter, not necessarily of the perfect-fluid form, dark energy, and a cosmological constant, and *all* field configurations of observed energy currents $J_k^{\varepsilon} \equiv T_{\hat{k}}^{\hat{0}}$. We followed this general approach already in our papers [3–5], which is in striking contrast to the other literature, which only treated the artificial situation of spherical shells of matter rotating rigidly around one given axis.

Linear cosmological perturbations can be decomposed into *scalar*, *vector*, and *tensor* sectors as discussed by Bardeen in 1980 [8]:

(1) In the *scalar sector*, 3-vector fields are gradients of scalar fields, the curl is zero, the fields are determined by their divergence. Traceless symmetric 3-tensors of second rank are obtained from scalar fields by 3-covariant derivatives.

(2) In the *vector sector*, 3-vector fields are divergenceless, given by the curl, i.e. *vorticity*. Therefore the vector sector is also called vorticity sector. Symmetric 3-tensor fields of second rank are obtained from vorticity vector fields by 3-covariant derivatives, and they are traceless but not divergenceless.

(3) In the *tensor sector*, traceless, divergenceless 3-tensor fields describe gravitational waves.

There is an important difference between our problem, the *angular momentum constraint*, and Bardeen's problem [8]: In our problem, the *position of the gyroscope* at P_0 is singled out. Relative to the gyroscope, the decomposition in eigenstates of *angular momentum* and *parity* and correspondingly the decomposition of the vector sector (\equiv vorticity sector) in *toroidal vorticity* versus *poloidal vorticity* is extremely useful:

(2a) *Toroidal vorticity* fields are defined to have *unnatural parity*, $P = (-1)^{\ell+1}$. Only this sector causes the *precession of gyroscopes* and *rotational dragging*. The simplest example of a toroidal vorticity field is the velocity field of a rotating shell of matter, which has $(\ell = 1, P = +1)$.

(2b) *Poloidal vorticity* fields have *natural parity*, $P = (-1)^\ell$. The simplest example of a poloidal vorticity field is the electric current in the wire wound around an iron ring.

We shall show that for our problem, the dragging of gyroscope's axes by cosmic energy currents, the mathematics of totally general linear perturbations fields is equivalent to the mathematics of the special case of spherical shells of matter at every radius around our selected observer with his gyroscopes, with every shell in rigid rotation around a different rotation axis. This is shown using three theorems, which are based on the *symmetries* relevant for Mach's principle, *rotation* and *parity*,

1. The precession of a gyroscope (relative to the local axes chosen by a given observer) *cannot* be caused by *scalar* perturbations nor, in linear perturbation theory, by *tensor* perturbations, because the energy-currents of gravitational waves are of second order in the gravitational field.
2. In the *vorticity sector* (\equiv vector sector), the precession of a gyroscope can be caused only by energy-current fields \vec{J}_ε with $J^P = 1^+$ relative to the given gyroscope's position, i.e. by *toroidal* vorticity and with $\ell = 1$.
3. On every mathematical spherical shell centered on the gyroscope considered: The energy-current field-

component which is toroidal and has $\ell = 1$ (relative to the gyroscope) is given by an *equivalent rigid rotation* with an *equivalent angular velocity* of matter $\vec{\Omega}_{\text{equiv}}^{\text{matter}}(\chi_s)$. The equivalent angular velocity of matter is given by the global inner product (scalar product) of the energy-current field \vec{J}_ε with the toroidal fields $\vec{X}_{\ell=1,m}^-$ on the shell of radius χ ,

$$\int d\Omega < \vec{X}_{\ell=1,m}^{-*}, \vec{J}_\varepsilon(\chi, \theta, \phi) > \equiv (J_\varepsilon)_{\ell=1,m}^-(\chi) \\ = -\sqrt{16\pi/3} (\rho_0 + p_0) R(\chi) [\Omega_m(\chi)]_{\text{equiv}}^{\text{matter}}, \quad (9)$$

where $< \dots, \dots >$ denotes the point-wise inner product, and $d\Omega$ is the element of solid angle, while Ω_m denotes spherical-basis components of the angular velocity. In the $m = 0$ sector, the energy current \vec{J}_ε is given by $J_\varepsilon^\phi = T^{t\phi} = aT^{\eta\phi} = aT^{v\phi}$.

The proofs of theorems (1) and (2) use the following facts: The precession of a gyroscope-spin $d\vec{S}/dt$ relative to given local axes, which equals the torque on the gyroscope, is an axial vector, $J^P = 1^+$. — For *scalar perturbations* all fields are derived from scalar fields via differentiation, but this can only produce source-fields \vec{J}_ε in the natural parity sequence, $0^+, 1^-, 2^+$, etc, which cannot contribute to the precession. — For *tensor perturbations* (gravitational waves), all linear perturbations are given by a traceless, divergenceless 3-tensor, from which one cannot form an axial vector field at the origin.

The proof of theorem (3) uses the following facts: For a general energy-momentum-stress tensor (not necessarily of the perfect-fluid type), the local center-of-mass 3-velocity \vec{v} in the toroidal vorticity sector with $m = 0$ is given by $v^\phi = (\rho_0 + p_0)^{-1} aT^{v\phi}$ in linear perturbation theory, where ρ_0 and p_0 refer to the unperturbed FRW background.

In the Green function, for a source at given χ_s , a toroidal velocity field with $\ell = 1$ is a flow equivalent to a rigid rotation of a shell of matter, and we shall choose the z -axis to be along the this shell-rotation axis, hence $m = 0$.

VI. MINKOWSKI CORRIDORS ALONG INCOMING WORLD-LINES OF PHOTONS

In this section, the discussion is *exact* (*non-perturbative*) and *without background geometry*. But we shall describe the procedure in a general cosmological language (no FRW background assumed).

We shall *define coordinates* x_P^μ for each *event* P by *measured* (observed) quantities. Therefore, *no gauge ambiguities* can arise.

We first choose an observation event P_0 anywhere in space-time. Next, we choose a *Local Orthonormal Basis* (LONB) at P_0 *fixed by measurements* (*observations*):

1. Choose an observer with his 4-velocity \bar{u} defining the time-like basis vector of the Local Ortho-Normal Basis (denoted by hats over indices),

$$\bar{e}_0(P_0) \equiv \bar{u}_{\text{observer}}(P_0).$$

We choose the *observer* to be at *rest* in the *asymptotic Hubble frame*, i.e. in the Hubble frame of quasars within some chosen fiducial volume at large luminosity distances $d_1 < d < d_2$ and over all angles. This means that the motion of the observer is chosen such that his observed dipole moment of radial quasar-velocities vanishes. This construction goes through, when the universe is far from a FRW universe.

2. The direction of arriving photons from a chosen “north-pole quasar” fixes the local basis vector $\bar{e}_z(P_0)$.
3. The direction of arriving photons from a chosen “zero-longitude quasar” fixes the *celestial null-meridian* $\phi = 0$ on the celestial sphere, which in turn fixes the local basis vector $\bar{e}_x(P_0)$.
4. The fourth basis-vector, $\bar{e}_{\hat{y}}$, must be Lorentz-orthogonal to the three other basis vectors.

This completely fixes the four basis vectors of our LONB(P_0) directly by cosmological observations.

We now fix the *coordinates* of each *event* on the past light-cone of P_0 , one radial and two angular coordinates, i.e. we uniquely fix the mapping

$$\text{event } P \Rightarrow x^\mu(P), \quad \mu = (1, 2, 3)$$

directly by measurements.

The angular coordinates (θ, ϕ) are constant (by construction) along every photon-world-line (geodesic) incoming at P_0 . The *event-coordinates* (θ_P, ϕ_P) are equal to the *arrival directions observed* at P_0 of photons emitted at the event P ,

$$\begin{aligned} & \text{emission-event coord. } (\theta_P, \phi_P)_{\text{emission}} \\ &= \text{observation angles } (\theta, \phi)_{\text{observed}}(P_0). \end{aligned}$$

For assigning a radial coordinate to every event P , an extremely useful concept is the *Minkowski corridor* with a choice of *Minkowski coordinates* along the world-line of one photon. We start from the local Minkowski coordinate system around the observation event P_0 , which is valid including first derivatives at P_0 of the metric $g_{\mu\nu}$. These Minkowski coordinates can be *extended along any one line* in space-time (geodesic or non-geodesic) as long as the line is not self-intersecting.

The book cover of Misner, Thorne, and Wheeler [9] shows the analogous concept of Euclidean corridors on the surface of an apple, and the first few pages of that book discuss this concept.

In our case, we extend the Minkowski coordinates of our observer at P_0 with his LONB in a Minkowski corridor along the world-line of one photon arriving at P_0

from an emission event P , and we denote the radial coordinate of the event P by r_P ,

$r_P \equiv$ radial distance of event in Minkowski coordinates of our chosen observer at P_0
along Minkowski corridor of world-line of photon.

For the astronomer, r_P is measured by the luminosity distance of an object, e.g. $r_P = 100$ Mpc. In classical electrodynamics [7], the spatial distance r_P of an event on the past light-cone is the integration variable in retarded potentials. — The *spatial separation* $(dr)_{PP'}$ must be distinguished from the *Lorentz-invariant space-time separation* $(ds^2)_{PP'}$, which is zero for events on a photon world-line. The 4-distance between two events is Lorentz-invariant, i.e. it is the same for all observers. In contrast the 3-distance depends on our chosen observer (resp. the output-observer in retarded potentials). The choice of an observer (at rest relative to asymptotic quasars) *induces* a *spatial metric* along any incoming photon world-line.

This completes the determination of the mapping from any event P to event-coordinates (r_P, θ_P, ϕ_P) directly by measurements.

Our procedure in this section has been in the spirit of defining Riemann normal coordinates in 3-space, which uses geodesics emerging from a point P_0 . Riemann normal coordinates refer to Riemannian space (purely spatial coordinates, no time), while our coordinates refer to the past light-cone.

The *apex point* of the past light-cone creates no difficulties in the retarded potentials of classical electromagnetism, and it creates no difficulties in our constraint equations.

The past light-cone, apart from the apex point P_0 , can be considered as a 3-dimensional (r, θ, ϕ) -space, a Riemannian 3-space. It is in this 3-space, where the *integration* in the *retarded potential* of our constraint equation will take place.

Three of the components of this 3-space Riemannian metric are given by the above construction,

$$\begin{aligned} {}^{(3)}g_{r\theta} &= 0, & {}^{(3)}g_{r\phi} &= 0, \\ {}^{(3)}g_{rr} &= 1. \end{aligned} \tag{10}$$

Proof of the orthogonality ${}^{(3)}g_{r\theta} = 0$ and ${}^{(3)}g_{r\phi} = 0$: At fixed r , we have a (θ, ϕ) -2-sphere, on which the radial distance from our observer at P_0 is independent of (θ, ϕ) . The orthogonality follows, because in a triangle with two equal sides r from P_0 to P and Q on the 2-sphere with an infinitesimal basis PQ (hence with an infinitesimal angle at the tip of the triangle), the angles at the basis of the triangle tend to $\pi/2$.

For formulating and solving the *angular constraint from the past light-cone*, it is *irrelevant*, whether the *universe* is *approximately FRW*:

1. In the angular momentum constraint, the input data of the transverse components of \vec{J}_ε on the past

light-cone are *averaged over all observation angles* using Eq. (9). Because of this angular averaging, it is *irrelevant*, whether the *universe* is *isotropic* around P_0 or not.

2. In the angular momentum constraint from the past light-cone, the input data \vec{J}_ε must also be *averaged* over all *radial distances* on the past light-cone with the weight function discussed at the end of this paper. — Even a FRW universe is *not radially homogeneous on the past light-cone*, e.g. the universe at redshift $z = 15$ looks very different from the universe at redshift $z = 0$.
3. At every radial distance, the *observed radial velocities* of matter must be *averaged over angles*. This gives an *expansion history* on the past light-cone. For a parametrization of this expansion history, one will use the expansion history given by the fits to FRW models given by WMAP and Planck.

VII. INNER GEOMETRY OF PAST LIGHT-CONE: UNCHANGED BY TOROIDAL VORTICITY WITH $\ell = 1$

The precession of a gyroscope at P_0 can be caused only by energy-current fields in the toroidal vorticity sector with $\ell = 1$, as shown in Sect. V.

With the coordinatization-mapping $P \Rightarrow x_P^\mu$ exactly fixed all over space-time in the previous section VI, the metric coefficients $g_{\mu\nu}$ are uniquely fixed by measurements.

The energy-current field of matter, the input, is in the toroidal vorticity sector with $\ell = 1$. For linear perturbations, symmetry arguments will force the geometric output $g_{\mu\nu}$ to be also in the same sector.

On the past light-cone of P_0 , we assume that the scalar sector is given by a spatially flat FRW background, and that the poloidal vorticity sector and the tensor sector are zero. Therefore the three components of the 4-metric, analogous to Eq. (10), are

$$\begin{aligned} {}^{(4)}g_{\chi\theta} &= 0, & {}^{(4)}g_{\chi\phi} &= 0, \\ {}^{(4)}g_{\chi\chi} &= 0. \end{aligned} \quad (11)$$

Toroidal vorticity with $\ell = 1$ can only generate *rigid rotations* on any 2-sphere. But rigid rotations leave the components of the *metric on the 2-spheres unchanged* from the FRW background (spatially flat) in Eqs. (3),

$$\text{including toroidal vorticity with } \ell = 1 : \\ g_{\theta\theta} = (a\chi)^2, \quad g_{\phi\phi} = (a\chi \sin \theta)^2, \quad g_{\theta\phi} = 0. \quad (12)$$

Conclusions:

1. For toroidal vorticity with $\ell = 1$ relative to a chosen observer, the *light-cones of this observer* have an *unperturbed inner geometry*, and the coordinates

can be chosen such that one has *unperturbed metric coefficients* $g_{\mu\nu}$ for the chosen light-cone coordinates (χ, θ, ϕ) .

2. The *inner geometry* of all light cones (vertex at any space-time point) remains *unperturbed* by toroidal vorticity with $\ell = 1$.

VIII. EVOLUTION FROM LIGHT CONE TO LIGHT CONE

A. The shift β

We now discuss the evolution from one light-cone to a neighbouring later light-cone (of our chosen observer) for the case of toroidal vorticity with $\ell = 1$.

The fundamental geometric quantity of this paper is the shift β .

For a usual (3+1)-split with *space-like hypersurface-slices* labelled by a time coordinate t , one defines the *shift* 3-vector-field and the *lapse* function by considering the *connector*-4-vectors \bar{C}_P , where (1) \bar{C}_P is *normal* on the slice through P , and (2) the 4-vector $(\bar{C}_P \delta t)$ connects slice $\Sigma_t(P)$ with the slice $\Sigma_{t+\delta t}$. See Misner, Thorne, and Wheeler [?].

For unperturbed FRW with a *fixed-Hubble-time slicing* and with the *conformal-time* coordinate η , the connector 4-vectors are $\bar{C}(P) = \bar{e}_\eta(P)$.

At first sight, there *seems* to be a *problem* with a (3+1)-split using light cones: The *normals* \bar{n} on *light-cones* are a multiple of \bar{e}_χ , i.e. \bar{n} lies in the tangent space to the light-cone and along a photon world-line. Therefore *normals on light cones do not connect* successive light-cones.

However, there is *no problem* for *toroidal vorticity* perturbations. In the *unperturbed* case, the natural choice for the *connectors* is $\bar{C} \equiv \bar{e}_v$, where $\bar{e}_v(P)\delta v$ connects our observer's past light cones with $v = v(P)$ and $v = v(P) + \delta v$. This connector field $\bar{C} = \bar{e}_v$, which connects light cones, is *identical* with the connector field $\bar{C} = \bar{e}_\eta$, which connects fixed-Hubble-conformal-time slices.

The *unperturbed lapse* function $N_0 \equiv \alpha_0$ is defined as the elapsed measured time τ (proper time) between light-cones along the connector, i.e. for $(\chi, \theta, \phi) = \text{fixed}$,

$$\text{unperturbed lapse} \equiv N_{(0)}(P) = (\partial\tau/\partial v)_P = a_P. \quad (13)$$

The *unperturbed shift* 3-vector $\vec{N}_{(0)}$ *vanishes*, because the unperturbed basis vector $\bar{e}_v(P) \equiv (\partial_v)_P$, the tangent vector to the unperturbed *v-coordinate line* is *not shifted away from* the unperturbed *connector*, $\bar{C}(P) = \bar{e}_v(P)$,

$$\text{unperturbed shift} \equiv \vec{N}_{(0)} = 0. \quad (14)$$

The notation N for the lapse function and N^i for the shift-3-vector is from Misner, Thorne, and Wheeler, Ref. [9].

Toroidal vorticity perturbations are in the 3-vector sector, therefore they cannot produce a lapse perturbation $N_{(1)}$, because the lapse function is a 3-scalar,

$$\text{lapse perturbation} \equiv N_{(1)} = 0. \quad (15)$$

Because *toroidal vorticity fields* with $m = 0$ point in the ϕ -direction, the perturbed connector \bar{C} can only acquire a ϕ component (in addition to the unperturbed v -component), and the *shift 3-vector* must point in the ϕ -direction. Therefore, all the action is in the (v, ϕ) -tangent space. The connector 4-vector $\bar{C}(P)$ and the shift-3-vector $\vec{N}(P)$ are defined by:

1. For infinitesimal δv , both $\bar{C}(P)\delta v$ and $\bar{e}_v(P)\delta v$ connect P with the neighboring coordinate line $v = v_P + \delta v$ in the tangent space to the (v, ϕ) -coordinate surface,

$$C^v = 1. \quad (16)$$

Within this (1+1)-dimensional (v, ϕ) -tangent space, the connector \bar{C}_P is defined to be *Lorentz-orthogonal* to $\bar{e}_\phi(P)$,

$$g(\bar{C}, \bar{e}_\phi) = 0. \quad (17)$$

2. At each point P , the *shift-3-vector* \vec{N} is defined as the difference between the 4-vectors \bar{e}_v (tangent to the v -coordinate line) and the connector \bar{C} ,

$$\vec{N} \equiv \bar{e}_v - \bar{C}, \quad (18)$$

i.e. \vec{N} has no v -component from property (1), and \vec{N} is the shift of \bar{e}_v relative to the (1+1)-normal \bar{C} , where we follow the *sign convention* of Misner, Thorne, and Wheeler [9].

3. With $m = 0$ for toroidal vorticity, the shift-3-vector \vec{N} can only have a ϕ -component, which we denote by $N^\phi \equiv \beta$,

$$m = 0 \Rightarrow \text{shift} \equiv \vec{N} = \beta \bar{e}_\phi \Rightarrow N^\phi = \beta. \quad (19)$$

A *positive shift*, $N^\phi > 0$, means that the *origin* of the ϕ -coordinate is *shifted relative to the connector* in the *positive direction* with a *shift angle per unit conformal time* $(d\phi/dv) = (d\phi/d\eta) = \beta$, hence

$$C^\phi = (d\phi/dv)_{(1+1)\text{normal}} = -N^\phi. \quad (20)$$

4. For $\ell = 1$ and $m = 0$, the shift is a *rigid rotation* of the (θ, ϕ) -coordinate system around the z -axis. Hence, the shift function $\beta(v, \chi)$ is independent of (θ, ϕ) . The shift function is the fundamental function for this paper.

The *lapse function* is defined as elapsed proper time τ per unit coordinate time v along the connector \bar{C} ,

$$\text{lapse} = (d\tau/dv)_{(1+1)\text{-normal}} \equiv N = a. \quad (21)$$

The angular velocity of a star measured by our observer at the origin is red-shifted from the value measured at the source. But the *angular change per unit conformal time* measured by our observer at P_0 is equal to the value measured locally at the source P ,

$$(d\phi/dv)_{\text{obs. at } P_0} = (d\phi/dv)_{\text{locally meas. at source}}.$$

B. The perturbation of the metric and the inverse metric

From now on, we shall denote unperturbed quantities by (0) and 1st-order perturbations by (1).

For toroidal vorticity perturbations with $\ell = 1$, the perturbed metric components $g_{\mu\nu}$ must have one index v , because perturbations only appear in the *evolution* from one light-cone to a neighboring light-cone, and one index ϕ , because the *shift* is in the ϕ -direction for $m = 0$. Hence, perturbations can only appear in $g_{v\phi}^{(1)}$.

The magnitude of the perturbation $g_{v\phi}$ follows from: (1) the connector \bar{C} is (1+1)-orthogonal to \bar{e}_ϕ , Eq. (17), (2) $C^v = 1$, Eq. (16), and (3) $C^\phi = -\beta$, Eq. (20),

$$\begin{aligned} 0 &= g(\bar{e}_\phi, \bar{C}) = g_{\phi v}^{(1)} C^v + g_{\phi\phi}^{(0)} C^\phi, \\ g_{v\phi}^{(1)} &= \beta g_{\phi\phi}^{(0)} = \beta a^2 \chi^2 \sin^2 \theta. \end{aligned} \quad (22)$$

The *line-element* for toroidal-vorticity perturbations with $(\ell = 1, m = 0)$ on a spatially flat FRW background in past-light-cone coordinates follows from the last equation and from Eq. (3),

$$\begin{aligned} ds^2 &= a^2(\eta)_{\eta=v-\chi} [-dv^2 + 2 dv d\chi \\ &\quad + \chi^2 (d\theta^2 + \sin^2 \theta d\phi^2) + 2\beta (\chi \sin \theta)^2 dv d\phi], \\ \beta &= \beta(v, \chi). \end{aligned} \quad (23)$$

To obtain the perturbation of the inverse metric one has to invert the (3x3)-matrix $g_{\mu\nu}$ in the (v, χ, ϕ) -sector, because the metric, Eq. (23), has no off-diagonal terms involving θ . The inversion gives only one perturbed matrix element,

$$g_{(1)}^{\chi\phi} = -\beta a^{-2}, \quad (24)$$

where it is useful to remember: (1) upper indices (ϕ, χ) , (2) minus sign, (3) a^{-2} multiplies β .

IX. MATTER INPUT MEASURABLE BEFORE SOLVING EINSTEIN'S EQUATIONS: $T^{v\phi}$

Einstein wrote in his letter to Felix Pirani of 2 February 1954 as quoted by Ehlers in [10]:

- “If you have a tensor $T_{\mu\nu}$ and not a metric, then this does not meaningfully describe matter. There is no theory of physics so far, which can describe matter without already the metric as a ingredient of

the description of matter. Therefore within existing theories the statement that the matter by itself determines the metric is neither wrong nor false, but it is meaningless.”

From this argument, Einstein drew the conclusion that “one should no longer speak of Mach’s principle at all”, quoted by Renn in [10].

Einstein’s argument is utterly important, but it is *half-correct* and *half-wrong*. It is our task to find out, *which component* of the energy-momentum tensor is (1) measurable before having solved Einstein’s equations, hence before knowing the metric components, and (2) relevant for our problem.

Should we consider an *upper* or *lower* ϕ -index to have a *directly measurable 3-momentum-input* on the matter side of Einstein’s equations? Should we consider an *upper* or *lower* v -index to have a *directly measurable density* on the *past light-cone*?

A. The angular velocity index: upper ϕ

For toroidal vorticity perturbations with $m = 0$, the 3-velocity of matter is in the ϕ -direction. Should we consider an *upper* or *lower* ϕ -index to have a *directly measurable 3-momentum-input* on the matter side of Einstein’s equations? 3-velocities and 4-velocities are prototypes for the geometric object *vector* in the narrow sense, for which the *natural* index-position is an *upper index*. A crucial observation was made for fixed t in Refs. [3–5]: The angular velocity ($d\phi/dt$) can be *directly measured* (for nearby stars). The 4-velocity is $u^\phi \approx d\phi/dt$ for non-relativistic motion of a star relative to the unperturbed Hubble flow. Conclusion:

- u^ϕ with the *upper index* ϕ is *locally measurable input*, the locally measured *angular velocity* of a star around the z -axis. Conclusion: For u^ϕ with the *upper index*, Einstein’s criticism is invalid.
- In contrast, u_ϕ , with a *lower index*, *cannot be used* as an *input* on the *matter-side* for *solving Einstein’s equations*, because in $u_\phi^{(1)} = g_{\phi\phi}^{(0)} u_{(1)}^\phi + g_{\phi v}^{(1)} u_{(0)}^v$ the metric perturbation $g_{\phi v}^{(1)}$ *cannot be known* as an *input* without having already solved Einstein’s equations all over the universe. Conclusion: For u_ϕ with the *lower index*, Einstein’s criticism is totally valid.
- Conclusion: The index for the angular velocity (and angular momentum) around the z -axis must be an *upper index* ϕ .

Before our papers [3–5], *all* papers on Mach’s principle missed the crucial *super-Hubble-radius suppression* of the dragging weight function, because they used a lower momentum-index.

Note: 3-velocities and 4-velocities are prototypes for the geometric object *vector* in the narrow sense, for which

the *natural, geometric* component-index-position is an *upper index*. *Without having solved Einstein’s equations*, one does not know the metric, and one *cannot pull down an upper index*.

B. The particle-density index: upper v

The *particle number* (e.g. baryons or galaxies) on the past light-cone within a given (χ, θ, ϕ) -coordinate domain is an observable, it is measured by astronomers. It is given by $n_{\alpha\beta\gamma}$, the *particle-density 3-form* in (χ, θ, ϕ) -coordinate space,

$$N_{\text{coord.domain}} = \int_{\text{coord.domain}} n_{\chi\theta\phi} d\chi d\theta d\phi. \quad (25)$$

Generally, a *3-form* is defined in coordinate-components as a tensor with *three antisymmetric lower indices*. An *integral* over a domain in three coordinates calls for a 3-form in this coordinate basis as an integrand *without weight factors*, as shown in Eq. (25). The *particle-density 3-form* $n_{\alpha\beta\gamma}$ gives the number of particles in the coordinate domain considered.

We use the 4-dimensional Levi-Civita tensor $\varepsilon_{\alpha\beta\gamma\delta}$, which is defined to be totally antisymmetric and to have $\varepsilon_{0123} = \varepsilon_{v\chi\theta\phi} \equiv +\sqrt{-\det({}^4g)} = a^4 \chi^2 \sin \theta$. Using $\varepsilon_{v\chi\theta\phi}$, we can convert the particle-density 3-form to a particle-density contravariant vector-component n^v with $n_{\chi\theta\phi} \equiv n^v \varepsilon_{v\chi\theta\phi} = n^v (a^4 \chi^2 \sin \theta)$, hence

$$N_{\text{coord.dom.}} = \int_{\text{coord.dom.}} (an^v) (a^3 \chi^2 \sin \theta) d\chi d\theta d\phi, \quad (26)$$

Conclusion:

- The index for particle-density on the past light-cone must be an *upper index* v .

The *source-input* for Einstein’s angular momentum constraint which is *measured* by our observer (with sufficiently precise apparatus in the future) is the *angular change of position per unit conformal time*, ($d\phi/dv$), and the corresponding *energy-current component* $T^{v\phi}$ with *two upper indices*, which gives, for non-relativistic peculiar velocities,

$$(\rho + p) (d\phi_{\text{matter}}/dv) = T^{v\phi} = aT^{t\phi} = aJ_\varepsilon^\phi.$$

$T^{v\phi}$ can be measured *without prior knowledge* of the *solution of Einstein’s equations* all over space-time, i.e. without prior knowledge of the metric field $g_{\mu\nu}$. It follows that we must consider the Einstein equation $G^{v\phi} = 8\pi G_N T^{v\phi}$. — All other components of the energy-momentum tensor $T^{\mu\nu}$ and of the Einstein tensor $G^{\mu\nu}$ are unperturbed.

Because the matter-source of the relevant constraint equation is the angular momentum, this constraint equation is called the *angular momentum constraint*.

X. EINSTEIN'S ANGULAR MOMENTUM CONSTRAINT

Apart from the FRW background, for toroidal vorticity with $(\ell = 1, m = 0)$, the Einstein-tensor has only the component $G^{v\phi}$, the energy-momentum has only the component $T^{v\phi}$, and the metric has only component $g_{v\phi}$.

The Einstein equation for $G^{v\phi}$ is,

$$G^{v\phi} = 8\pi G_N T^{v\phi}.$$

The computation of the exact $G^{v\phi}$ from the exact $g_{v\phi}$ for toroidal vorticity with $(\ell = 1, m = 0)$ using standard methods will be documented in an appendix in an extended version of this paper. The result with the sign convention of Misner, Thorne, and Wheeler [9] is,

$$\begin{aligned} a^4 G^{v\phi} &= \\ &= \partial_\chi^2 \beta / 2 + (\partial_\chi \beta)(2\chi^{-1} - \mathcal{H}) + \beta(2\mathcal{H}' - 2\mathcal{H}^2). \end{aligned} \quad (27)$$

In the energy-momentum tensor, $T^{\mu\nu} = (\rho + p)u^\mu u^\nu$, we consider *non-relativistic peculiar velocities* of vorticity flows relative to the FRW background. This implies that the perturbations are linear. — For toroidal vorticity with $(\ell = 1, m = 0)$, the energy-momentum tensor has only one non-zero component different from the FRW background (denoted by a subscript zero),

$$a^2 T^{v\phi} = (\rho + p)_{(0)} (d\phi/dv)_{\text{matter}}. \quad (28)$$

\mathcal{H}^2 and \mathcal{H}' are given in terms of ρ and p for a spatially flat FRW background,

$$\begin{aligned} \mathcal{H}^2 &= a^2 (8\pi/3) G_N \rho, \\ \mathcal{H}' &= -a^2 (4\pi/3) G_N (\rho + 3p), \\ \mathcal{H}' - \mathcal{H}^2 &= -a^2 4\pi G_N (\rho + p). \end{aligned} \quad (29)$$

XI. OBSERVER ROTATING RELATIVE TO UNPERTURBED FRW UNIVERSE

In this section, we specialize to an unperturbed FRW universe. There are no vorticity fields.

We no longer fix the orientation of the local spatial axes of the observer at P_0 to the observed directions to asymptotic quasars. Instead, we fix the *orientation* of the *local spatial axes* of the *observer* by *two local landmarks*.

For a consistency test, we assume that the observer (with his local ortho-normal basis) is *rotating relative* to the *FRW universe* around his local z -axis with angular velocity,

$$\begin{aligned} \Omega_{\text{observer rel. to universe}} &= -\Omega_{\text{matter rel. to observer}} \\ &= -a^{-1} (d\phi/dv)_{\text{matter}}. \end{aligned}$$

On the geometric side of Einstein's $G^{v\phi}$ equation, the shift function $\beta \equiv -(d\phi/dv)_{(1+1)\text{normal}}$ is independent of χ , and the $G^{v\phi}$ equation reduces to

$$\begin{aligned} G_{(1)}^{v\phi} &\Rightarrow -2a^{-2}\beta(4\pi G_N)(\rho + p), \\ (8\pi G_N)T_{(1)}^{v\phi} &= a^{-2}(8\pi G_N)(\rho + p)(d\phi/dv)_{\text{matter}}, \end{aligned}$$

hence, relative to the observer we have,

$$\beta \equiv (d\phi/dv)_{(1+1)\text{normal}} = (d\phi/dv)_{\text{matter}}. \quad (30)$$

This result proves *exact dragging of inertial axes* by matter in an *unperturbed FRW universe*: If all matter in the universe rotates rigidly around the observer, then the *gyroscope axes* (at the position of the observer) are *exactly dragged* by the rotating matter.

This result is highly non-trivial: this result would not hold for general relativity e.g. in a universe with the observed galaxies out to redshift $z = 1000$ and no matter beyond.

As discussed in Sect. IX of [4], Einstein's equations together with given matter sources $T^{\mu\nu}$ are insufficient to obtain the geometry in asymptotic Minkowski space. General relativity (without explicit and totally non-trivial boundary conditions) is not invariant under going to a rotating coordinate system.

XII. ORDINARY LINEAR DIFFERENTIAL EQUATION FROM VORTICITY WITH $\ell = 1$

In this section, we consider the source-free, *homogeneous* differential equation $G^{v\phi} = 0$ from Eq. (27). This source-free equation will be needed for the *Green function* away from the rotating spherical source-shell at χ_{source} .

This equation can be solved numerically for any history of the Hubble rate.

For either a *cold-matter dominated* (CMD) universe or a *radiation dominated* (RD) universe, the homogeneous equation Eq. (27) becomes particularly simple,

$$\begin{aligned} a(\eta) &= \eta^P, \\ \text{observation event:} & \quad a_0 \equiv 1, \quad \eta_0 \equiv 1, \\ \text{big bang:} & \quad a_{\text{BB}} = 0, \quad \eta_{\text{BB}} = 0, \\ \text{matter dominated:} & \quad P = 2, \\ \text{radiation dominated:} & \quad P = 1, \\ \mathcal{H} &= P\eta^{-1}, \quad \mathcal{H}' = -P\eta^{-2}, \end{aligned}$$

On the light-cone of the observer, it is advantageous to use as the independent variable the conformal time η instead of the comoving distance χ ,

$$\text{light-cone of observation at } P_0 : \quad \chi = 1 - \eta.$$

The homogeneous Einstein equation, $G^{v\phi} = 0$, needed for the Green function away from the thin source-shell, becomes,

$$\partial_\eta^2 \beta - (\partial_\eta \beta)[4(1 - \eta)^{-1} - 2P\eta^{-1}] - 4\beta(P + P^2)\eta^{-2} = 0. \quad (31)$$

This ordinary linear differential equation of second order for the shift $\beta(\eta)$ has *three singular points* of the *regular type*, conventionally called “regular singular points”.

The *definition* of a “regular singular point”: With a prefactor 1 for the second derivative β'' , the *prefactor* for

the *first derivative* β' has at most a *single pole*, and the *prefactor* for the *function* β has at most a *double pole*, [11, 12].

The important *result*: At *regular* singular points, the solutions have at most *algebraic singularities*, $\beta(\eta) \propto \eta^\alpha$.

Any ordinary linear differential equation of second order with three regular singular points is in the class of *Riemann's differential equation* [11, 12]. Our differential equation for $\beta(\eta)$ has three regular singular points at $\eta = 0$ (big bang), $\eta = 1$ (observation event), and $\eta = \infty$ (infinite future).

The *exponents* of $\beta(\eta)$ for our differential equation are,

$$\begin{aligned} \text{big bang,} \quad \eta \rightarrow 0 : \quad & \beta \rightarrow \eta^\alpha, \\ (\alpha, \alpha') = & -P + 1/2 \pm \sqrt{5P^2 + 3P + 1/4}, \\ \text{obs. event,} \quad (\eta - 1) \rightarrow 0 : \quad & \beta \rightarrow (\eta - 1)^{\bar{\beta}}, \\ (\bar{\beta}, \bar{\beta}') = & (0, -3), \\ \text{inf. future,} \quad (1/\eta) \rightarrow 0 : \quad & \beta \rightarrow (1/\eta)^\gamma, \\ (\gamma, \gamma') = & P + 3/2 \pm \sqrt{5P^2 + 7P + 9/4}. \end{aligned} \quad (32)$$

We have denoted Riemann's exponents at the observation point by $(\bar{\beta}, \bar{\beta}')$ to distinguish them from our shift function $\beta(\eta)$. The sum of all six exponents must be equal to one $\alpha + \alpha' + \bar{\beta} + \bar{\beta}' + \gamma + \gamma' = 1$ for all Riemann differential equations.

At the observation event, the exponents are the same as in Minkowski space. At the big bang, one exponent gives a *power-law suppression* in the Green function for β , to be compared with the exponential suppression for large distances found on a slice of fixed Hubble-time in [3–5].

The regular solution at the observation event, $\eta = 1$, goes to one, and is given by hypergeometric series. The solution decaying towards the big bang, after division by β^α , also goes to one, and it is given by another hypergeometric series series.

Although we have not yet obtained the numerical solution for the Green function, a cold-matter dominated or a radiation-dominated universe, nor for the more realistic universe with dark energy plus cold dark matter according to WMAP and Planck, it is clear that the numerical solution exists and is well behaved.

From the Green function, one directly and simply obtains the dragging weight function as in [3–5],

From our analysis at fixed Hubble time [3–5], we expect that most of the dragging is done by matter around redshift $z \approx 1$.

XIII. PROOF OF MACH'S HYPOTHESIS FOR LINEAR PERTURBATIONS OF FRW

The *Green function* $G_\beta(\chi, \chi')$ for the shift β is obtained by solving Einstein's homogeneous $G^{v\phi}$ equation, as

discussed in the last section. This Green function, taken for χ infinitesimally close to the observation event P_0 at $\chi = 0$, gives directly the precession rate of the gyroscope at $\chi = 0$ due to a rotating source-shell at $\chi' = \chi_{\text{source}}$ as discussed in [4, 5]. In other words: this gives the *weight function for dragging*, $W(\chi)$, obtained for a fixed-Hubble-time slice in [4, 5].

The weight function $W(\chi)$ for dragging by energy-currents on the past light-cone depends on the history of the Hubble rate in the universe. The dragging weight function is peaked near $z_{\text{source}} = 1$ relative to the observation [3–5]. For an observation time at today's redshift ≈ 20 , a good approximation is a *matter dominated* universe. For an observation time at today's redshift $\approx 10'000$, a good approximation is a *radiation dominated* universe. For observations today, a good approximation is a universe *dominated by cold matter plus dark energy*.

The dragging weight function for these three histories of the Hubble rate will be evaluated numerically in a subsequent paper. For a matter-dominated or a radiation-dominated universe, this involves nothing more than various hypergeometric series.

The resulting graphs will be instructive. But the *explicit forms* of the various dragging weight functions are *not needed* for a *general proof* of *exact dragging* of inertial axes, i.e. the proof of the hypothesis formulated by Mach.

The crucial observation:

$$\begin{aligned} & \text{from Eq. 30 follows:} \\ & \text{dragging weight function } W(\chi) \text{ normalized to unity,} \\ & \int_{\text{observation}}^{\text{big bang}} d\chi W(\chi) = 1. \end{aligned} \quad (33)$$

From this fact, Mach's principle follows directly,

$$\begin{aligned} \vec{\Omega}_{\text{gyro}}(P_0) &= < \vec{\Omega}_{\text{matter}} >_{W(\chi)}^{\text{past light cone}} \\ &= \int_{\text{observation}}^{\text{big bang}} d\chi W(\chi) \vec{\Omega}_{\text{matter}}(\chi). \end{aligned} \quad (34)$$

Conclusions:

- The *hypothesis formulated by Ernst Mach* has been proved for all *linear* perturbations on the *past light-cone* of the observation on spatially flat FRW backgrounds.
- The solution of the *angular momentum constraint* from the *past light-cone* gives *nothing more and nothing less* than (1) the proof of *exact dragging* of *inertial axes* by *cosmic energy currents*, (2) the form of the *dragging weight-functions* for various Hubble-rate histories.

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